The Geometry of the Moduli Space of Curves of Genus 23

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1 Introduction

The problem of describing the birational geometry of the moduli space $\mathcal{M}_g$ of complex curves of genus g has a long history. Severi already knew in 1915 that $\mathcal{M}_g$ is unirational for $g \leq 10$ (cf. [Sev]; see also [AC1] for a modern proof). In the same paper Severi conjectured that $\mathcal{M}_g$ is unirational for all genera g. Then for a long period this problem seemed intractable (Mumford writes in [Mu], p.51:"Whether more $\mathcal{M}_g$'s, $g \geq 11$, are unirational or not is a very interesting problem, but one which looks very hard too, especially if g is quite large"). The breakthrough came in the eighties when Eisenbud, Harris and Mumford proved that $\mathcal{M}_g$ is of general type as soon as $g \geq 24$ and that the Kodaira dimension of $\mathcal{M}_{23}$ is ≥ 1 (see [HM], [EH3]). We note that $\mathcal{M}_g$ is rational for $g \leq 6$ (see [Dol] for problems concerning the rationality of various moduli spaces).

Severi's proof of the unirationality of $\mathcal{M}_g$ for small g was based on representing a general curve of genus g as a plane curve of degree d with δ nodes; this is possible when $d \geq 2g/3+2$. When the number of nodes is small, i.e. $\delta \leq (d+1)(d+2)/6$, the dominant map from the variety of plane curves of degree d and genus g to $\mathcal{M}_g$ yields a rational parametrization of the moduli space. The two conditions involving d and δ can be satisfied only when $g \leq 10$, so Severi's argument cannot be extended for other genera. However, using much more subtle ideas, Chang, Ran and Sernesi proved the unirationality of $\mathcal{M}_g$ for $g = 11, 12, 13$ (see [CR1], [Se1]), while for $g = 15, 16$ they proved that the Kodaira dimension is $-\infty$ (see [CR2,4]). The remaining cases $g = 14$ and $17 \leq g \leq 23$ are still quite mysterious. Harris and Morrison conjectured in [HMo] that $\mathcal{M}_g$ is uniruled precisely when $g < 23$.

All these facts indicate that $\mathcal{M}_{23}$ is a very interesting transition case. Our main result is the following:

Theorem 1 *The Kodaira dimension of the moduli space of curves of genus* 23 *is* ≥ 2.

We will also present some evidence for the hypothesis that the Kodaira dimension of $\mathcal{M}_{23}$ is actually equal to 2.

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2 Multicanonical linear systems and the Kodaira dimension of $\mathcal{M}_g$

We study three multicanonical divisors on $\mathcal{M}_{23}$, which are (modulo some boundary components) of Brill-Noether type, and we conclude by looking at their relative position that $\kappa(\mathcal{M}_{23}) \geq 2$.

We review some notations. We shall denote by $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{C}}_g$ the moduli spaces of stable and 1-pointed stable curves of genus g over $\mathbb{C}$. If C is a smooth algebraic curve of genus g, we consider for any r and d, the scheme whose points are the $\mathfrak{g}^r_d$'s on C, that is,

$$G^r_d(C) = \{(\mathcal{L}, V) : \mathcal{L} \in \text{Pic}^{\,d}(C), V \subseteq H^0(C, \mathcal{L}), \dim(V) = r+1\},$$

(cf. [ACGH]) and denote the associated Brill-Noether locus in $\mathcal{M}_g$ by

$$\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g : G^r_d(C) \neq \emptyset\},$$

and by $\overline{\mathcal{M}}^r_{g,d}$ its closure in $\overline{\mathcal{M}}_g$.

The distribution of linear series on algebraic curves is governed (to some extent) by the *Brill-Noether number*

$$\rho(g, r, d) := g - (r+1)(g - d + r).$$

The Brill-Noether Theorem asserts that when $\rho(g, r, d) \geq 0$ every curve of genus g possesses a $\mathfrak{g}^r_d$, while when $\rho(g, r, d) < 0$ the general curve of genus g has no $\mathfrak{g}^r_d$'s, hence in this case the Brill-Noether loci are proper subvarieties of $\mathcal{M}_g$. When $\rho(g, r, d) < 0$, the naive expectation that $-\rho(g, r, d)$ is the codimension of $\mathcal{M}^r_{g,d}$ inside $\mathcal{M}_g$, is in general way off the mark, since there are plenty of examples of Brill-Noether loci of unexpected dimension (cf. [EH2]). However, we have Steffen's result in one direction (see [St]):

If $\rho(g, r, d) < 0$ then each component of $\mathcal{M}^r_{g,d}$ has codimension at most $-\rho(g, r, d)$ in $\mathcal{M}_g$.

On the other hand, when the Brill-Noether number is not very negative, the Brill-Noether loci tend to behave nicely. Existence of components of $\mathcal{M}^r_{g,d}$ of the expected dimension has been proved for a rather wide range, namely for those g, r, d such that $\rho(g, r, d) < 0$, and

$$\rho(g, r, d) \geq \begin{cases} -g + r + 3 & \text{if } r \text{ is odd;} \\ -rg/(r+2) + r + 3 & \text{if } r \text{ is even.} \end{cases}$$

We have a complete answer only when $\rho(g, r, d) = -1$. Eisenbud and Harris have proved in [EH2] that in this case $\mathcal{M}^r_{g,d}$ has a unique divisorial component, and using the previously mentioned theorem of Steffen's, we obtain the following result:

If $\rho(g, r, d) = -1$, then $\overline{\mathcal{M}}^r_{g,d}$ is an irreducible divisor of $\overline{\mathcal{M}}_g$.

We will also need Edidin's result (see [Ed2]) which says that for $g \geq 12$ and $\rho(g, r, d) = -2$, all components of $\mathcal{M}^r_{g,d}$ have codimension 2. We can get codimension 1 Brill-Noether conditions only for the genera g for which $g + 1$ is composite. In that case we can write

$$g + 1 = (r + 1)(s - 1), \ s \geq 3$$

and set $d := rs - 1$. Obviously $\rho(g, r, d) = -1$ and $\overline{\mathcal{M}}^r_{g,d}$ is an irreducible divisor. Furthermore, its class has been computed (cf. [EH3]):

$$[\overline{\mathcal{M}}^r_{g,d}] = c_{g,r,d}\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i\right),$$

where $c_{g,r,d}$ is a positive rational number equal to $3\mu/(2g-4)$, with μ being the number of $\mathfrak{g}^r_d$'s on a general pointed curve (C_0, q) of genus $g-2$ with ramification sequence $(0, 1, 2, \ldots, 2)$ at q. For $g = 23$ we have the following possibilities:

$$(r, s, d) = (1, 13, 12),\ (11, 3, 32),\ (2, 9, 17),\ (7, 4, 24),\ (3, 7, 20),\ (5, 5, 24).$$

It is immediate by Serre duality, that cases $(1, 13, 12)$ and $(11, 3, 32)$ yield the same divisor on $\mathcal{M}_{23}$, namely the 12-gonal locus $\mathcal{M}^1_{12}$; similarly, cases $(2, 9, 17)$ and $(7, 4, 24)$ yield the divisor $\mathcal{M}^2_{17}$ of curves having a $\mathfrak{g}^2_{17}$, while cases $(3, 7, 20)$ and $(5, 5, 24)$ give rise to $\mathcal{M}^3_{20}$, the divisor of curves having a $\mathfrak{g}^3_{20}$. Note that when the genus we are referring to is clear from the context, we write $\mathcal{M}^r_d = \mathcal{M}^r_{g,d}$.

By comparing the classes of the Brill-Noether divisors to the class of the canonical divisor $K_{\overline{\mathcal{M}}_{g,reg}} = 13\lambda - 2\delta_0 - 3\delta_1 - \cdots - 2\delta_{[g/2]}$, at least in the case when $g+1$ is composite we can infer that

$$K_{\overline{\mathcal{M}}_{g,reg}} = a[\overline{\mathcal{M}}^r_{g,d}] + b\lambda + (\text{ positive combination of } \delta_0, \ldots, \delta_{[g/2]}),$$

where a is a positive rational number, while $b > 0$ as long as $g \geq 24$ but $b = 0$ for $g = 23$. As it is well-known that λ is big on $\overline{\mathcal{M}}_g$, it follows that $\mathcal{M}_g$ is of general type for $g \geq 24$ and that it has non-negative Kodaira dimension when $g = 23$. Specifically for $g = 23$, we get that there are positive integer constants m, m_1, m_2, m_3 such that:

$$mK = m_1[\overline{\mathcal{M}}^1_{12}] + E,\ mK = m_2[\overline{\mathcal{M}}^2_{17}] + E,\ mK = m_3[\overline{\mathcal{M}}^3_{20}] + E, \tag{1}$$

where E is the same positive combination of $\delta_1, \ldots, \delta_{11}$.

Proposition 2.1 (Eisenbud-Harris, [EH3]) *There exists a smooth curve of genus* 23 *that possesses a* $\mathfrak{g}^1_{12}$ *, but no* $\mathfrak{g}^2_{17}$*. It follows that* $\kappa(\mathcal{M}_{23}) \geq 1$.

Harris and Mumford proved (cf. [HM]) that $\overline{\mathcal{M}}_g$ has only canonical singularities for $g \geq 4$, hence $H^0(\overline{\mathcal{M}}_{g,reg}, nK) = H^0(\widetilde{\mathcal{M}}_g, nK)$ for each $n \geq 0$, with $\widetilde{\mathcal{M}_g}$ a desingularization of $\overline{\mathcal{M}}_g$. We already know that $\dim(\mathrm{Im}\phi_{mK}) \geq 1$, where $\phi_{mK} : \overline{\mathcal{M}}_{23} \dashrightarrow \mathbb{P}^\nu$ is the multicanonical map, m being as in (1). We will prove that $\kappa(\mathcal{M}_{23}) \geq 2$. Indeed, let us assume that $\dim(\mathrm{Im}\phi_{mK}) = 1$. Denote by $C := \overline{\mathrm{Im}\phi_{mK}}$ the Kodaira image of $\overline{\mathcal{M}}_{23}$. We reach a contradiction by proving two things:

- α) The Brill-Noether divisors $\mathcal{M}^1_{12}, \mathcal{M}^2_{17}$ and $\mathcal{M}^3_{20}$ are mutually distinct.
- β) There exist smooth curves of genus 23 which belong to exactly two of the Brill-Noether divisors from above.

This suffices in order to prove Theorem 1: since $\overline{\mathcal{M}}^1_{12}$, $\overline{\mathcal{M}}^2_{17}$ and $\overline{\mathcal{M}}^3_{20}$ are part of different

multicanonical divisors, they must be contained in different fibres of the multicanonical map ϕ_{mK}. Hence there exists different points $x, y, z \in C$ such that

$$\mathcal{M}^1_{12} = \overline{\phi^{-1}(x)} \cap \mathcal{M}_{23},\ \mathcal{M}^2_{17} = \overline{\phi^{-1}(y)} \cap \mathcal{M}_{23},\ \mathcal{M}^3_{20} = \overline{\phi^{-1}(z)} \cap \mathcal{M}_{23}.$$

It follows that the set-theoretic intersection of any two of them will be contained in the base locus of $|mK_{\overline{\mathcal{M}}_{23}}|$. In particular:

$$\mathrm{supp}(\mathcal{M}^1_{12}) \cap \mathrm{supp}(\mathcal{M}^2_{17}) = \mathrm{supp}(\mathcal{M}^2_{17}) \cap \mathrm{supp}(\mathcal{M}^3_{20}) = \mathrm{supp}(\mathcal{M}^3_{20}) \cap \mathrm{supp}(\mathcal{M}^1_{12}), \quad (2)$$

and this contradicts β). We complete the proof of α) and β) is Section 5.

3 Deformation theory for $\mathfrak{g}^r_d$'s and limit linear series

We recall a few things about the variety parametrising $\mathfrak{g}^r_d$'s on the fibres of the universal curve (cf. [AC2]), and then we recap on the theory of limit linear series (cf. [EH1], [Mod]), which is our main technique for the study of $\mathcal{M}_{23}$.

Given g, r, d and a point $[C] \in \mathcal{M}_g$, there is a connected neighbourhood U of $[C]$, a finite ramified covering $h : \mathcal{M} \to U$, such that $\mathcal{M}$ is a fine moduli space of curves (i.e. there exists $\xi : \mathcal{C} \to \mathcal{M}$ a universal curve), and a proper variety over $\mathcal{M}$,

$$\pi : \mathcal{G}^r_d \to \mathcal{M}$$

which parametrizes classes of couples (C, l), with $[C] \in \mathcal{M}$ and $l \in G^r_d(C)$, where we have made the identification $C = \xi^{-1}([C])$.
Let (C, l) be a point of $\mathcal{G}^r_d$ corresponding to a curve C and a linear series $l = (\mathcal{L}, V)$, where $\mathcal{L} \in \mathrm{Pic}^d(C), V \subseteq H^0(C, \mathcal{L})$, and $\dim(V) = r + 1$. By choosing a basis in V, one has a morphism $f : C \to \mathbb{P}^r$. The normal sheaf of f is defined through the exact sequence

$$0 \longrightarrow T_C \longrightarrow f^*(T_{\mathbb{P}^r}) \longrightarrow N_f \longrightarrow 0. \quad (3)$$

By dividing out the torsion of N_f one gets to the exact sequence

$$0 \longrightarrow \mathcal{K}_f \longrightarrow N_f \longrightarrow N'_f \longrightarrow 0, \quad (4)$$

where the torsion sheaf $\mathcal{K}_f$ (the cuspidal sheaf) is based at those points $x \in C$ where $df(x) = 0$, and N'_f is locally free of rank $r - 1$. The tangent space $T_{(C,l)}(\mathcal{G}^r_d)$ fits into an exact sequence (cf. [AC2]):

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{Hom}(V, V) \longrightarrow H^0(C, N_f) \longrightarrow T_{(C,l)}(\mathcal{G}^r_d) \longrightarrow 0, \quad (5)$$

from which we have that $\dim T_{(C,l)}(\mathcal{G}^r_d) = 3g - 3 + \rho(g, r, d) + h^1(C, N_f)$.

Proposition 3.1 *Let C be a curve and $l \in G^r_d(C)$ a base point free linear series. Then the variety $\mathcal{G}^r_d$ is smooth and of dimension $3g-3+\rho(g, r, d)$ at the point (C, l) if and only if $H^1(C, N_f) = 0$.*

Remark: The condition $H^1(C, N_f) = 0$ is automatically satisfied for $r = 1$ as N_f is a sheaf with finite support. Thus $\mathcal{G}^1_d$ is smooth of dimension $2g+2d-5$. It follows that $\mathcal{G}^1_d$ is birationally equivalent to the d-gonal locus $\mathcal{M}^1_d$ when $d < (g+2)/2$.

Limit linear series try to answer questions of the following kind: what happens to a family of $\mathfrak{g}^r_d$'s when a smooth curve specializes to a reducible curve? Limit linear series solve such problems for a class of reducible curves, those of compact type. A curve C is of *compact type* if the dual graph is a tree. A curve C is *tree-like* if, after deleting edges leading from a node to itself, the dual graph becomes a tree.

Let C be a smooth curve of genus g and $l = (\mathcal{L}, V) \in G^r_d(C)$, $\mathcal{L} \in \mathrm{Pic}^d(C), V \subseteq H^0(C, \mathcal{L})$, and $\dim(V) = r+1$. Fix $p \in C$ a point. By ordering the finite set $\{ord_p(\sigma)\}_{\sigma\in V}$ one gets the *vanishing sequence* of l at p:

$$a^l(p) : 0 \leq a^l_0(p) < \ldots < a^l_r(p) \leq d.$$

The *ramification sequence* of l at p

$$\alpha^l(p) : 0 \leq \alpha^l_0(p) \leq \ldots \leq \alpha^l_r(p) \leq d - r$$

is defined as $\alpha^l_i(p) = a^l_i(p) - i$ and the *weight* of l at p is

$$w^l(p) = \sum_{i=0}^{r} \alpha^l_i(p).$$

A *Schubert index of type* (r, d) is a sequence of integers $\beta : 0 \leq \beta_0 \leq \ldots \beta_r \leq d - r$. If α and β are Schubert indices of type (r, d) we write $\alpha \leq \beta \iff \alpha_i \leq \beta_i, i = 0, \ldots, r$. The point p is said to be a *ramification point* of l if $w^l(p) > 0$. The linear series l is said to have a *cusp* at p if $\alpha^l(p) \geq (0, 1, \ldots, 1)$. For C a tree-like curve, $p_1, \ldots, p_n \in C$ smooth points and $\alpha^1, \ldots, \alpha^n$ Schubert indices of type (r, d), we define

$$G^r_d(C, (p_1, \alpha^1), \ldots (p_n, \alpha^n)) := \{l \in G^r_d(C) : \alpha^l(p_1) \geq \alpha^1, \ldots, \alpha^l(p_n) \geq \alpha^n\}.$$

This scheme can be realized naturally as a determinantal variety and its expected dimension is

$$\rho(g, r, d, \alpha^1, \ldots, \alpha^n) := \rho(g, r, d) - \sum_{i=1}^{n} \sum_{j=0}^{r} \alpha^i_j.$$

If C is a curve of compact type, a *crude limit* $\mathfrak{g}^r_d$ on C is a collection of ordinary linear series $l = \{l_Y \in G^r_d(Y) : Y \subseteq C \text{ is a component }\}$, satisfying the following compatibility condition: if Y and Z are components of C, with $\{p\} = Y \cap Z$, then

$$a^{l_Y}_i(p) + a^{l_Z}_{r-i}(p) \geq d, \text{ for } i = 0, \ldots r.$$

If equality holds everywhere, we say that l is a *refined limit* $\mathfrak{g}^r_d$. The 'honest' linear series $l_Y \in G^r_d(Y)$ is called the Y-aspect of the limit linear series l.

We will often use the additivity of the Brill-Noether number: if C is a curve of compact

type, for each component $Y \subseteq C$, let $q_1, \dots, q_s$ be the points where Y meets the other components of C. Then for any limit $\mathfrak{g}^r_d$ on C we have the following inequality:

$$\rho(g, r, d) \geq \sum_{Y \subseteq C} \rho(l_Y, \alpha^{l_Y}(q_1), \dots, \alpha^{l_Y}(q_s)), \tag{6}$$

with equality if and only if l is a refined limit linear series.

It has been proved in [EH1] that limit linear series arise indeed as limits of ordinary linear series on smooth curves. Suppose we are given a family $\pi : \mathcal{C} \to B$ of genus g curves, where $B = \mathrm{Spec}(R)$ with R a complete discrete valuation ring. Assume furthermore that $\mathcal{C}$ is a smooth surface and that if $0, \eta$ denote the special and generic point of B respectively, the central fibre C_0 is reduced and of compact type, while the generic geometric fibre C_η is smooth and irreducible. If $l_\eta = (\mathcal{L}_\eta, V_\eta)$ is a $\mathfrak{g}^r_d$ on X_η, there is a canonical way to associate a crude limit series l_0 on C_0 which is the limit of l_η in a natural way: for each component Y of C_0, there exists a unique line bundle $\mathcal{L}^Y$ on $\mathcal{C}$ such that

$$\mathcal{L}^Y_{|\mathcal{C}_\eta} = \mathcal{L}_\eta \text{ and } \deg_Z(\mathcal{L}^Y_{|Z}) = 0,$$

for any component Z of C_0 with $Z \neq Y$. (This implies of course that $\deg_Y(\mathcal{L}^Y_{|Y}) = d$). Define $V^Y = V_\eta \cap H^0(\mathcal{C}, \mathcal{L}^Y) \subseteq H^0(\mathcal{C}_\eta, \mathcal{L}_\eta)$. Clearly, V^Y is a free R-module of rank $r+1$. Moreover, the composite homomorphism

$$V^Y(0) \to (\pi_* \mathcal{L}^Y)(0) \to H^0(C_0, \mathcal{L}^Y_{|C_0}) \to H^0(Y, \mathcal{L}^Y_{|C_0})$$

is injective, hence $l_Y = (\mathcal{L}^Y_{|Y}, V^Y(0))$ is an ordinary $\mathfrak{g}^r_d$ on Y. One proves that $l = \{l_Y : Y \text{ component of } C_0\}$ is a limit linear series.

If C is a reducible curve of compact type, l a limit $\mathfrak{g}^r_d$ on C, we say that l is *smoothable* if there exists $\pi : \mathcal{C} \to B$ a family of curves with central fibre $C = C_0$ as above, and $(\mathcal{L}_\eta, V_\eta)$ a $\mathfrak{g}^r_d$ on the generic fibre C_η whose limit on C (in the sense previously described) is l.

Remark: If a stable curve of compact type C, has no limit $\mathfrak{g}^r_d$'s, then $[C] \notin \overline{\mathcal{M}}^r_{g,d}$. If there exists a smoothable limit $\mathfrak{g}^r_d$ on C, then $[C] \in \overline{\mathcal{M}}^r_{g,d}$.

Now we explain a criterion due to Eisenbud and Harris (cf. [EH1]), which gives a sufficient condition for a limit $\mathfrak{g}^r_d$ to be smoothable. Let l be a limit $\mathfrak{g}^r_d$ on a curve C of compact type. Fix $Y \subseteq C$ a component, and $\{q_1, \dots, q_s\} = Y \cap \overline{(C - Y)}$. Let

$$\pi : \mathcal{Y} \to B, \ \tilde{q}_i : B \to \mathcal{Y}$$

be the versal deformation space of $(Y, q_1, \dots q_s)$. The base B can be viewed as a small $(3g(Y) - 3 + s)$-dimensional polydisk. Using general theory one constructs a proper scheme over B,

$$\sigma : \mathcal{G}^r_d(\mathcal{Y}/B; (\tilde{q}_i, \alpha^{l_Y}(q_i))_{i=1}^s) \to B$$

whose fibre over each $b \in B$ is $\sigma^{-1}(b) = G^r_d(Y_b, (\tilde{q}_i(b), \alpha^{l_Y}(q_i))_{i=1}^s)$. One says that l is *dimensionally proper with respect to* Y, if the Y-aspect l_Y is contained in some component $\mathcal{G}$ of $\mathcal{G}^r_d(\mathcal{Y}/B; (\tilde{q}_i, \alpha^{l_Y}(q_i))_{i=1}^s)$ of the expected dimension, i.e.

$$\dim \mathcal{G} = \dim B + \rho(l_Y, \alpha^{l_Y}(q_1), \dots \alpha^{l_Y}(q_s)).$$

One says that l is *dimensionally proper*, if it is dimensionally proper with respect to any component $Y \subseteq C$. The 'Regeneration Theorem' (cf. [EH1]) states that every dimensionally proper limit linear series is smoothable.

The next result is a 'strong Brill-Noether Theorem', i.e. it not only asserts a Brill-Noether type statement, but also singles out the locus where the statement fails.

Proposition 3.2 (Eisenbud-Harris) *Let C be a tree-like curve and for any component $Y \subseteq C$, denote by $q_1, \ldots, q_s \in Y$ the points where Y meets the other components of C. Assume that for each Y the following conditions are satisfied:*

a. If $g(Y) = 1$ then $s = 1$.

b. If $g(Y) = 2$ then $s = 1$ and q is not a Weierstrass point.

c. If $g(Y) \geq 3$ then $(Y, q_1, \ldots, q_s)$ is a general s-pointed curve.

Then for $p_1, \ldots p_t \in C$ general points, $\rho(l, \alpha^l(p_1), \ldots, \alpha^l(p_t)) \geq 0$ for any limit linear series on C.

Simple examples involving pointed elliptic curves show that the condition $\rho(g, r, d) \geq \sum_{i=1}^t w^l(p_i)$ does not guarantee the existence of a linear series $l \in G^r_d(C)$ with prescribed ramification at general points $p_1, p_2, \ldots, p_t \in C$. The appropriate condition in the pointed case can be given in terms of Schubert cycles. Let $\alpha = (\alpha_0, \ldots, \alpha_r)$ be a Schubert index of type (r, d) and

$$\mathbb{C}^{d+1} = W_0 \supset W_1 \supset \ldots \supset W_{d+1} = 0$$

a decreasing flag of linear subspaces. We consider the Schubert cycle in the Grassmanian,

$$\sigma_\alpha = \{\Lambda \in G(r+1, d+1) : \dim(\Lambda \cap W_{\alpha_i + i}) \geq r + 1 - i, \ i = 0, \ldots, r\}.$$

For a general t-pointed curve $(C, p_1, \ldots, p_t)$ of genus g, and $\alpha^1, \ldots, \alpha^t$ Schubert indices of type (r, d), the necessary and sufficient condition that C has a $\mathfrak{g}^r_d$ with ramification α^i at p_i is that

$$\sigma_{\alpha^1} \cdot \ldots \cdot \sigma_{\alpha^t} \cdot \sigma^g_{(0,1,\ldots,1)} \neq 0 \text{ in } H^*(G(r+1, d+1), \mathbb{Z}). \tag{7}$$

In the case $t = 1$ this condition can be made more explicit (cf. [EH3]): a general pointed curve (C, p) of genus g carries a $\mathfrak{g}^r_d$ with ramification sequence $(\alpha_0, \ldots, \alpha_r)$ at p, if and only if

$$\sum_{i=0}^r (\alpha_i + g - d + r)_+ \leq g, \tag{8}$$

where $x_+ = \max\{x, 0\}$. One can make the following simple but useful observation:

Proposition 3.3 *Let (C, p, q) be a general 2-pointed curve of genus g and $(\alpha_0, \ldots, \alpha_r)$ a Schubert index of type (r, d). Then C has a $\mathfrak{g}^r_d$ having ramification sequence $(\alpha_0, \ldots, \alpha_r)$ at p and a cusp at q if and only if*

$$\sum_{i=0}^r (\alpha_i + g + 1 - d + r)_+ \leq g + 1.$$

Proof: The condition for the existence of the $\mathfrak{g}^r_d$ with ramification α at p and a cusp at q is that $\sigma_\alpha \cdot \sigma^{g+1}_{(0,1,\dots,1)} \neq 0$ (cf. (7)). According to the Littlewood-Richardson rule (see [F]), this is equivalent with $\sum_{i=0}^r (\alpha_i + g + 1 - d + r)_+ \leq g + 1$. □

4 A few consequences of limit linear series

We investigate the Brill-Noether theory of a 2-pointed elliptic curve (see also [EH4]), and we prove that $\overline{\mathcal{M}}^r_{g,d} \cap \Delta_1$ is irreducible for $\rho(g,r,d) = -1$.

Proposition 4.1 *Let (E, p, q) be a two-pointed elliptic curve. Consider the sequences*

$$a: \ a_0 < a_1 < \dots a_r \leq d, \quad b: \ b_0 < b_1 < \dots b_r \leq d.$$

1. For any linear series $l = (\mathcal{L}, V) \in G^r_d(E)$ one has that $\rho(l, \alpha^l(p), \alpha^l(q)) \geq -r$. Furthermore, if $\rho(l, \alpha^l(p), \alpha^l(q)) \leq -1$, then $p - q \in Pic^0(E)$ is a torsion class.
2. Assume that the sequences a and b satisfy the inequalities: $d - 1 \leq a_i + b_{r-i} \leq d, \ i = 0, \dots, r$. Then there exists at most one linear series $l \in G^r_d(E)$ such that $a^l(p) = a$ and $a^l(q) = b$. Moreover, there exists exactly one such linear series $l = (\mathcal{O}_E(D), V)$ with $D \in E^{(d)}$, if and only if for each $i = 0, \dots, r$ the following is satisfied: if $a_i + b_{r-i} = d$, then $D \sim a_i \ p + b_{r-i} \ q$, and if $(a_i + 1) \ p + b_{r-i} \ q \sim D$, then $a_{i+1} = a_i + 1$.

Proof: In order to prove 1. it is enough to notice that for dimensional reasons there must be sections $\sigma_i \in V$ such that $\text{div}(\sigma_i) \geq a^l_i(p) \ p + a^l_{r-i}(q) \ q$, therefore, $a^l_i(p) + a^l_{r-i}(q) \leq d$. By adding up all these inequalities, we get that $\rho(l, \alpha^l(p), \alpha^l(q)) \geq -r$. Furthermore, $\rho(l, \alpha^l(p), \alpha^l(q)) \leq -1$ precisely when for at least two values $i < j$ we have equalities $a_i + b_{r-i} = d, \ a_j + b_{r-j} = d$, which means that there are sections $\sigma_i, \sigma_j \in V$ such that $\text{div}(\sigma_i) = a_i \ p + b_{r-i} \ q, \ \text{div}(\sigma_j) = a_j \ p + b_{r-j} \ q$. By subtracting, we see that $p - q \in \text{Pic}^0(E)$ is torsion. The second part of the Proposition is in fact Prop.5.2 from [EH4]. □

Proposition 4.2 *Let g, r, d be such that $\rho(g,r,d) = -1$. Then the intersection $\overline{\mathcal{M}}^r_{g,d} \cap \Delta_1$ is irreducible.*

Proof: Let Y be an irreducible component of $\overline{\mathcal{M}}^r_{g,d} \cap \Delta_1$. Either $Y \cap \text{Int}\Delta_1 \neq \emptyset$, hence $Y = \overline{Y \cap \text{Int}\Delta_1}$, or $Y \subseteq \Delta_1 - \text{Int}\Delta_1$. The second alternative never occurs. Indeed, if $Y \subseteq \Delta_1 - \text{Int}\Delta_1$, then since codim $(Y, \overline{\mathcal{M}}_g) = 2$, Y must be one of the irreducible components of $\Delta_1 - \text{Int}\Delta_1$. The components of $\Delta_1 - \text{Int}\Delta_1$ correspond to curves with two nodes. We list these components (see [Ed1]):

- For $1 \leq j \leq g - 2$, Δ_{1j} is the closure of the locus in $\overline{\mathcal{M}}_g$ whose general point corresponds to a chain composed of an elliptic curve, a curve of genus $g - j - 1$, and a curve of genus j.
- The component Δ_{01}, whose general point corresponds to the union of a smooth elliptic curve and an irreducible nodal curve of genus $g - 2$.
- The component $\Delta_{0,g-1}$ whose general point corresponds to the union of a smooth curve of genus $g - 1$ and an irreducible rational curve.

As the general point of $\Delta_{1,j}, \Delta_{0,1}$ or $\Delta_{0,g-1}$ is a tree-like curve which satisfies the conditions of Prop.3.2 it follows that such a curve satisfies the 'strong' Brill-Noether Theorem, hence $\Delta_{1,j} \nsubseteq \overline{\mathcal{M}}^r_{g,d}$, $\Delta_{0,1} \nsubseteq \overline{\mathcal{M}}^r_{g,d}$ and $\Delta_{0,g-1} \nsubseteq \overline{\mathcal{M}}^r_{g,d}$, a contradiction. So, we are left with the first possibility: $Y = \overline{Y \cap \mathrm{Int}\Delta_1}$. We are going to determine the general point $[C] \in Y \cap \mathrm{Int}\Delta_1$. Let $X = C \cup E, g(C) = g-1, E$ elliptic, $E \cap C = \{p\}$ such that X carries a limit $\mathfrak{g}^r_d$, say l. By the additivity of the Brill-Noether number, we have:

$$-1 = \rho(g,r,d) \geq \rho(l,C,p) + \rho(l,E,p).$$

Since $\rho(l,E,p) \geq 0$, it follows that $\rho(l,C,p) \leq -1$, so $w^{l_C}(p) \geq r$. Let us denote by

$$\beta : \mathcal{C}_{g-1} \times \mathcal{C}_1 \to \mathrm{Int}\Delta_1$$

the natural map given by $\beta([C,p],[E,q]) = [X := C \cup E/p \sim q]$. We claim that if we choose X generically, then $\alpha_0^{l_C}(p) = 0$. If not, p is a base point of l_C and after removing the base point we get that $[C] \in \mathcal{M}^r_{g-1,d-1}$. Note that $\rho(g-1,r,d-1) = -2$, so $\dim \mathcal{M}^r_{g-1,d-1} = 3g-8$ (cf. [Ed2]). If we denote by $\pi : \mathcal{C}_{g-1} \to \mathcal{M}_{g-1}$ the morphism which 'forgets the point', we get that

$$\dim\, \beta(\pi^{-1}(\mathcal{M}^r_{g-1,d-1}) \times \mathcal{C}_1) = 3g-6 < \dim\, Y,$$

a contradiction. Hence, for the generic $[X] \in Y$ we must have $\alpha_0^{l_C}(p) = 0$, so $a_r^{l_E}(p) = d$. Since an elliptic curve cannot have a meromorphic function with a single pole, it follows that $a_{r-1}^{l_E}(p) \leq d-2$ and this implies $\alpha^{l_C}(p) \geq (0,1,\ldots,1)$, i.e. l_C has a cusp at p. Thus, if we introduce the notation

$$\mathcal{C}^r_{g-1,d}(0,1,\ldots,1) = \{[C,p] \in \mathcal{C}_{g-1} : G^r_d(C,(p,(0,1,\ldots,1))) \neq \emptyset\},$$

then $Y \subseteq \overline{\beta(\mathcal{C}^r_{g-1,d}(0,1.\ldots,1) \times \mathcal{C}_1)}$. On the other hand, it is known (cf. [EH2]) that $\mathcal{C}^r_{g-1,d}(0,1,\ldots,1)$ is irreducible of dimension $3g-6$ (that is, codimension 1 in $\mathcal{C}_{g-1}$), so we must have $Y = \overline{\beta(\mathcal{C}^r_{g-1,d}(0,1,\ldots,1) \times \mathcal{C}_1)}$, which not only proves that $\overline{\mathcal{M}}^r_{g,d} \cap \Delta_1$ is irreducible, but also determines the intersection. □

5 The Kodaira dimension of $\mathcal{M}_{23}$

In this section we prove that $\kappa(\mathcal{M}_{23}) \geq 2$ and we investigate closely the multicanonical linear systems on $\overline{\mathcal{M}}_{23}$. We now describe the three multicanonical Brill-Noether divisors from Section 2.

5.1 The divisor $\overline{\mathcal{M}}^1_{12}$

There is a stratification of $\mathcal{M}_{23}$ given by gonality:

$$\mathcal{M}^1_2 \subseteq \mathcal{M}^1_3 \subseteq \ldots \subseteq \mathcal{M}^1_{12} \subseteq \mathcal{M}_{23}.$$

For $2 \leq d \leq g/2+1$ one knows that $\mathcal{M}^1_k = \mathcal{M}^1_{g,k}$ is an irreducible variety of dimension $2g+2d-5$. The general point of $\mathcal{M}^1_{g,d}$ corresponds to a curve having a unique $\mathfrak{g}^1_d$.

5.2 The divisor $\overline{\mathcal{M}}^2_{17}$

The *Severi variety* $V_{d,g}$ of irreducible plane curves of degree d and geometric genus g, where $0 \leq g \leq \binom{d-1}{2}$, is an irreducible subscheme of $\mathbb{P}^{d(d+3)/2}$ of dimension $3d+g-1$ (cf. [H], [Mod]). Inside $V_{d,g}$ we consider the open dense subset $U_{d,g}$ of irreducible plane curves of degree d having exactly $\delta = \binom{d-1}{2} - g$ nodes and no other singularities. There is a global normalization map

$$m : U_{d,g} \to \mathcal{M}_g, \ m([Y]) := [\tilde{Y}], \ \tilde{Y} \text{ is the normalization of } Y.$$

When $d-2 \leq g \leq \binom{d-1}{2}, d \geq 5$, $U_{d,g}$ has the expected number of moduli, i.e.

$$\dim m(U_{d,g}) = \min(3g-3, 3g-3+\rho(g,2,d)).$$

In our case we can summarize this as follows:

Proposition 5.1 *There is exactly one component of $\mathcal{G}^2_{17}$ mapping dominantly to $\mathcal{M}^2_{17}$. The general element $(C,l) \in \mathcal{G}^2_{17}$ corresponds to a curve C of genus 23, together with a $\mathfrak{g}^2_{17}$ which provides a plane model for C of degree 17 with 97 nodes.*

5.3 The divisor $\overline{\mathcal{M}}^3_{20}$

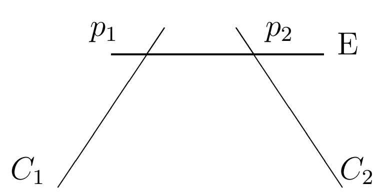
Here we combine the result of Eisenbud and Harris (see [EH2]) about the uniqueness of divisorial components of $\mathcal{G}^r_d$ when $\rho(g,r,d) = -1$, with Sernesi's (see [Se2]) which asserts the existence of components of the Hilbert scheme $H_{d,g}$ parametrizing curves in $\mathbb{P}^3$ of degree d and genus g with the expected number of moduli, for $d-3 \leq g \leq 3d-18, d \geq 9$.

Proposition 5.2 *There is exactly one component of $\mathcal{G}^3_{20}$ mapping dominantly to $\mathcal{M}^3_{20}$. The general point of this component corresponds to a pair (C,l) where C is a curve of genus 23 and l is a very ample $\mathfrak{g}^3_{20}$.*

We are going to prove that the Brill-Noether divisors $\overline{\mathcal{M}}^1_{12}, \overline{\mathcal{M}}^2_{17}$ and $\overline{\mathcal{M}}^3_{20}$ are mutually distinct.

Theorem 2 *There exists a smooth curve of genus 23 having a $\mathfrak{g}^2_{17}$, but no $\mathfrak{g}^3_{20}$'s. Equivalently, one has* $\text{supp}(\mathcal{M}^2_{17}) \nsubseteq \text{supp}(\mathcal{M}^3_{20})$.

Proof: It suffices to construct a reducible curve X of compact type of genus 23, which has a smoothable limit $\mathfrak{g}^2_{17}$, but no limit $\mathfrak{g}^3_{20}$. If $[C] \in \mathcal{M}_{23}$ is a nearby smoothing of X which preserves the $\mathfrak{g}^2_{17}$, then $[C] \in \mathcal{M}^2_{17} - \mathcal{M}^3_{20}$. Let us consider the following curve:



$$X := C_1 \cup C_2 \cup E,$$

where (C_1, p_1) and (C_2, p_2) are general pointed curves of genus 11, E is an elliptic curve, and $p_1 - p_2$ is a primitive 9-torsion point in $\mathrm{Pic}^0(E)$

Step 1) *There is no limit* $\mathfrak{g}^3_{20}$ *on* X. Assume that l is a limit $\mathfrak{g}^3_{20}$ on X. By the additivity of the Brill-Noether number,

$$-1 \geq \rho(l_{C_1}, p_1) + \rho(l_{C_2}, p_2) + \rho(l_E, p_1, p_2).$$

Since (C_i, p_i) are general points in $\mathcal{C}_{11}$, it follows from Prop.3.2 that $\rho(l_{C_i}, p_i) \geq 0$, hence $\rho(l_E, p_1, p_2) \leq -1$. On the other hand $\rho(l_E, p_1, p_2) \geq -3$ from Prop.4.1.

Denote by (a_0, a_1, a_2, a_3) the vanishing sequence of l_E at p_1, and by (b_0, b_1, b_2, b_3) that of l_E at p_2. The condition (8) for a general pointed curve $[(C_i, p_i)] \in \mathcal{C}_{11}$ to possess a $\mathfrak{g}^3_{20}$ with prescribed ramification at the point p_i and the compatibility conditions between l_{C_i} and l_E at p_i give that:

$$(14 - a_3)_+ + (13 - a_2)_+ + (12 - a_1)_+ + (11 - a_0)_+ \leq 11, \tag{9}$$

and

$$(14 - b_3)_+ + (13 - b_2)_+ + (12 - b_1)_+ + (11 - b_0)_+ \leq 11. \tag{10}$$

1st case: $\rho(l_E, p_1, p_2) = -3$. Then $a_i + b_{3-i} = 20$, for $i = 0, \ldots, 3$ and it immediately follows that $20(p_1 - p_2) \sim 0$ in $\mathrm{Pic}^0(E)$, a contradiction.
2nd case: $\rho(l_E, p_1, p_2) = -2$. We have two distinct possibilities here: i) $a_0 + b_3 = 20, a_1 + b_2 = 20, a_2 + b_1 = 20, a_3 + b_0 = 19$. Then it follows that $a^{l_E}(p_1) = (0, 9, 18, 19)$ and $a^{l_E}(p_2) = (0, 2, 11, 20)$, while according to (9), $a_3 \leq 15$, (because $\rho(l_{C_1}, p_1) \leq 1$), a contradiction. ii) $a_0 + b_3 = 20, a_1 + b_2 = 20, a_2 + b_1 = 19, a_3 + b_0 = 20$. Again, it follows that $a_3 = a_0 + 18 \geq 15$, a contradiction.
3rd case: $\rho(l_E, p_1, p_2) = -1$. Then $\rho(l_{C_i}, p_i) = 0$ and l is a refined limit $\mathfrak{g}^3_{20}$. From (9) and (10) we must have: $a^{l_E}(p_i) \leq (11, 12, 13, 14)$, $i = 1, 2$. There are four possibilities: i) $a_0 + b_3 = a_1 + b_2 = 20, a_2 + b_1 = a_3 + b_0 = 19$. Then $a_1 = a_0 + 9 \leq 12$, so $b_3 = 20 - a_0 \geq 17$, a contradiction. ii) $a_0 + b_3 = a_2 + b_1 = 20, a_2 + b_1 = a_3 + b_0 = 19$. Then $b_3 = 20 - a_0 \leq 14$, so $a_2 = a_0 + 9 \geq 15$, a contradiction. iii) $a_0 + b_3 = a_3 + b_0 = 20, a_1 + b_2 = a_2 + b_1 = 19$. Then $b_3 = 19 - a_0 \leq 14$, so $a_3 \geq a_0 + 9 \geq 15$, a contradiction. iv) $a_0 + b_3 = a_3 + b_0 = 19, a_1 + b_2 = a_2 + b_1 = 20$. Then $b_3 = 19 - a_0 \leq 14$, so $a_2 \geq a_1 + 9 \geq 15$, a contradiction again. We conclude that X has no limit $\mathfrak{g}^3_{20}$.

Step 2) *There exists a smoothable limit* $\mathfrak{g}^2_{17}$ *on* X*, hence* $[X] \in \overline{\mathcal{M}}^2_{17}$. We construct a limit linear series l of type $\mathfrak{g}^2_{17}$ on X, aspect by aspect: on C_i take $l_{C_i} \in G^2_{17}(C_i)$ such that $a^{l_{C_i}}(p_i) = (4, 9, 13)$. Note that in this case $\sum_{j=0}^{r}(\alpha_j + g - d + r)_+ = g$, so (8) ensures the existence of such a $\mathfrak{g}^2_{17}$. On E we take $l_E = |V_E|$, where $|V_E| \subseteq |4p_1 + 13p_2| = |4p_2 + 13p_1|$ is a $\mathfrak{g}^2_{17}$ with vanishing sequence $(4, 8, 13)$ at p_i. Prop.4.1 ensures the existence of such a linear series. In this way l is a refined limit $\mathfrak{g}^2_{17}$ on X with $\rho(l_{C_i}, p_i) = 0, \rho(l_E, p_1, p_2) = -1$. We prove that l is dimensionally proper. Let $\pi_i : \mathcal{C}_i \to \Delta_i$, $\tilde{p}_i : \Delta_i \to \mathcal{C}_i$, be the versal deformation of $[(C_i, p_i)] \in \mathcal{C}_{11}$, and $\sigma_i : \mathcal{G}^2_{17}(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, (4, 8, 11))) \to \Delta_i$ the projection.

Since being general is an open condition, we have that σ_i is surjective and dim $\sigma_i^{-1}(t) = \rho(l_{C_i}, p_i) = 0$, for each $t \in \Delta_i$, therefore

$$\dim \mathcal{G}^2_{17}(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, (4, 8, 11))) = \dim \Delta_i + \rho(l_{C_i}, p_i) = 31.$$

Next, let $\pi : \mathcal{C} \to \Delta$, $\tilde{p}_1, \tilde{p}_2 : \Delta \to \mathcal{C}$ be the versal deformation of (E, p_1, p_2). We prove that

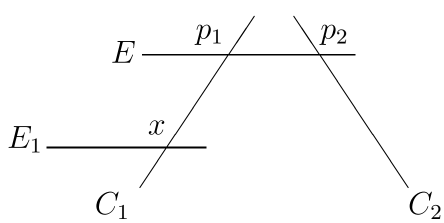
$$\dim \mathcal{G}^2_{17}(\mathcal{C}/\Delta, (\tilde{p}_i, (4, 7, 11))) = \dim \Delta + \rho(l_E, p_1, p_2) = 1.$$

This follows from Prop.4.1, since a 2-pointed elliptic curve $(E_t, \tilde{p}_1(t), \tilde{p}_2(t))$ has at most one $\mathfrak{g}^2_{17}$ with ramification $(4, 7, 11)$ at both $\tilde{p}_1(t)$ and $\tilde{p}_2(t)$, and exactly one when $9(\tilde{p}_1(t) - \tilde{p}_2(t)) \sim 0$. Hence $\mathrm{Im}\mathcal{G}^2_{17}(\mathcal{C}/\Delta, (\tilde{p}_i, (4, 7, 11))) = \{t \in \Delta : 9(\tilde{p}_1(t) - \tilde{p}_2(t)) \sim 0 \text{ in } \mathrm{Pic}^0(E_t)\}$, which is a divisor on Δ, so the claim follows and l is a dimensionally proper $\mathfrak{g}^2_{17}$. $\square$

A slight variation of the previous argument gives us:

Proposition 5.3 *We have* $\mathrm{supp}(\overline{\mathcal{M}}^2_{17} \cap \Delta_1) \neq \mathrm{supp}(\overline{\mathcal{M}}^3_{20} \cap \Delta_1)$.

*P*roof: We construct a curve $[Y] \in \Delta_1 \subseteq \overline{\mathcal{M}}_{23}$ which has a smoothable limit $\mathfrak{g}^2_{17}$ but no limit $\mathfrak{g}^3_{20}$. Let us consider the following curve:



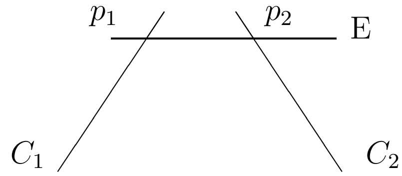
$$Y := C_1 \cup C_2 \cup E_1 \cup E,$$

where (C_2, p_2) is a general point of $\mathcal{C}_{11}$, (C_1, p_1, x) is a general 2-pointed curve of genus 10, (E_1, x) is general in $\mathcal{C}_1$, E is an elliptic curve, and $p_1 - p_2 \in \mathrm{Pic}^0(E)$ is a primitive 9-torsion. In order to prove that Y has no limit $\mathfrak{g}^3_{20}$, one just has to take into account that according to Prop.3.3, the condition for a general 1-pointed curve (C, z) of genus g, to have a $\mathfrak{g}^r_d$ with ramification α at z is the same with the condition for a general 2-pointed curve (D, x, y) of genus $g - 1$ to have a $\mathfrak{g}^r_d$ with ramification α at x and a cusp at y. Therefore we can repeat what we did in the proof of Theorem 2. Next, we construct l, a smoothable limit $\mathfrak{g}^2_{17}$ on Y: take $l_{C_2} \in G^2_{17}(C_2, (p_2, (4, 8, 11))), l_E = |V_E| \subseteq |4p_1 + 13p_2|$, with $\alpha^{l_E}(p_i) = (4, 7, 11)$, on E_1 take $l_{E_1} = 14x + |3x|$, and finally on C_1 take l_{C_1} such that $\alpha^{l_{C_1}}(p_1) = (4, 8, 11), \alpha^{l_{C_1}}(x) = (0, 0, 1)$. Prop.3.3 ensures the existence of l_{C_1}. Clearly, l is a refined limit $\mathfrak{g}^2_{17}$ and the proof that it is smoothable is all but identical to the one in the last part of Theorem 2. $\square$

The other cases are settled by the following:

Theorem 3 *There exists a smooth curve of genus* 23 *having a* $\mathfrak{g}^1_{12}$ *but having no* $\mathfrak{g}^2_{17}$ *nor* $\mathfrak{g}^3_{20}$. *Equivalently,* $\mathrm{supp}(\mathcal{M}^1_{12}) \nsubseteq \mathrm{supp}(\mathcal{M}^2_{17})$ *and* $\mathrm{supp}(\mathcal{M}^1_{12}) \nsubseteq \mathrm{supp}(\mathcal{M}^3_{20})$.

Proof: We take the curve considered in [EH3]:



$$Y := C_1 \cup C_2 \cup E,$$

where (C_i, p_i) are general points of $\mathcal{C}_{11}$, E is elliptic and $p_1 - p_2 \in \mathrm{Pic}^0(E)$ is a primitive 12-torsion. Clearly Y has a (smoothable) limit $\mathfrak{g}^1_{12}$: on C_i take the pencil $|12p_i|$, while on E take the pencil spanned by $12p_1$ and $12p_2$. It is proved in [EH3] that Y has no limit $\mathfrak{g}^2_{17}$'s and similarly one can prove that Y has no limit $\mathfrak{g}^3_{20}$'s either. We omit the details.$\Box$

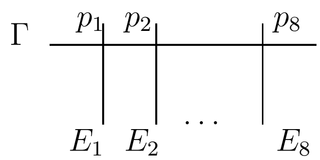
Now we are going to prove that equation (2)

$$\mathrm{supp}(\mathcal{M}^1_{12}) \cap \mathrm{supp}(\mathcal{M}^2_{17}) = \mathrm{supp}(\mathcal{M}^2_{17}) \cap \mathrm{supp}(\mathcal{M}^3_{20}) = \mathrm{supp}(\mathcal{M}^3_{20}) \cap \mathrm{supp}(\mathcal{M}^1_{12})$$

is impossible, and as explained before, this will imply that $\kappa(\mathcal{M}_{23}) \geq 2$. The main step in this direction is the following:

Proposition 5.4 *There exists a stable curve of compact type of genus* 23 *which has a smoothable limit* $\mathfrak{g}^3_{20}$*, a smoothable limit* $\mathfrak{g}^2_{15}$ *(therefore also a* $\mathfrak{g}^2_{17}$*), but has generic gonality, that is, it does not have any limit* $\mathfrak{g}^1_{12}$*.*

Proof We shall consider the following stable curve X of genus 23:



$$X := \Gamma \cup E_1 \cup \ldots \cup E_8,$$

where the E_i's are elliptic curves, $\Gamma \subseteq \mathbb{P}^2$ is a general smooth plane septic and the points of attachment $\{p_i\} = \Gamma \cup E_i$ are general points of Γ.

Step 1) *There is no limit* $\mathfrak{g}^1_{12}$ *on* X. Assume that l is a limit $\mathfrak{g}^1_{12}$ on X. Since the elliptic curves E_i cannot have meromorphic functions with a single pole, we have that $a^{l_{E_i}}(p_i) \leq (10, 12)$, hence $\alpha^{l_\Gamma}(p_i) \geq (0, 1)$, that is, l_Γ has a cusp at p_i for $i = 1, \ldots, 8$. We now prove that Γ has no $\mathfrak{g}^1_{12}$'s with cusps at the points p_i.

First, we notice that $\dim G^1_{12}(\Gamma) = \rho(15, 1, 12) = 7$. Indeed, if we assume that $\dim G^1_{12}(\Gamma) \geq 8$, by applying Keem's Theorem (cf. [ACGH], p.200) we would get that Γ possesses a $\mathfrak{g}^1_4$, which is impossible since $\mathrm{gon}(\Gamma) = 6$. (In general, if $Y \subseteq \mathbb{P}^2$ is a smooth plane curve, $\deg(Y) = d$, then gon $(Y) = d - 1$, and the $\mathfrak{g}^1_{d-1}$ computing the gonality is cut out by the lines passing through a point $p \in Y$, see [ACGH].) Next, we define the variety

$$\Sigma = \{(l, q_1, \ldots, q_8) \in G^1_{12}(\Gamma) \times \Gamma^8 : \alpha^l(q_i) \geq (0, 1), i = 1, \ldots, 8\}$$

and denote by $\pi_1 : \Sigma \to G^1_{12}(\Gamma)$ and $\pi_2 : \Sigma \to \Gamma^8$ the two projections. For any $l \in G^1_{12}(\Gamma)$, the fibre $\pi_1^{-1}(l)$ is finite, hence dim $\Sigma =$ dim $G^1_{12}(\Gamma) = 7$, which shows that π_2 cannot be surjective and this proves our claim.

Step 2) *There exists a smoothable limit* $\mathfrak{g}^2_{17}$ *on* X, *hence* $[X] \in \overline{\mathcal{M}}^2_{17}$. We construct l, a limit $\mathfrak{g}^2_{17}$ on X as follows: on Γ there is a (unique) $\mathfrak{g}^2_7$, and we consider $l_\Gamma = \mathfrak{g}^2_7(p_1+\cdots+p_8)$, i.e. the $\Gamma-$ aspect l_Γ is obtained from the $\mathfrak{g}^2_7$ by adding the base points $p_1, \ldots, p_8$. Clearly $a^{l_\Gamma}(p_i) = (1,2,3)$ for each i. On E_i we take $l_{E_i} = \mathfrak{g}^2_3(12p_i)$ for $i = 1, \ldots, 8$, where $\mathfrak{g}^2_3$ is a complete linear series of the form $|2p_i + x_i|$, with $x_i \in E_i - \{p_i\}$. Furthermore, $a^{l_{E_i}}(p_i) = (12, 13, 14)$, so $l = \{l_\Gamma, l_{E_i}\}$ is a refined limit $\mathfrak{g}^2_{15}$ on X. One sees that $\rho(l_{E_i}, \alpha^{l_{E_i}}(p_i)) = 1$ for all i, $\rho(l_\Gamma, \alpha^{l_\Gamma}(p_1), \ldots, \alpha^{l_\Gamma}(p_8)) = -15$, and $\rho(l) = -7$. We now prove that l is dimensionally proper.

Let $\pi_i : \mathcal{C}_i \to \Delta_i$, $\tilde{p}_i : \Delta_i \to \mathcal{C}_i$ be the versal deformation space of (E_i, p_i), for $i = 1, \ldots, 8$. There is an obvious isomorphism over Δ_i

$$\mathcal{G}^2_{15}(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, (12,12,12))) \simeq \mathcal{G}^2_3(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, 0)).$$

If $\sigma_i : \mathcal{G}^2_3(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, 0)) \to \Delta_i$ is the natural projection, then for each $t \in \Delta_i$, the fibre $\sigma_i^{-1}(t)$ is isomorphic to $\pi_i^{-1}(t)$, the isomorphism being given by

$$\pi_i^{-1}(t) \ni q \mapsto |2\tilde{p}_i(t) + q| \in G^2_3(\pi_i^{-1}(t)).$$

Thus, $\mathcal{G}^2_3(\mathcal{C}_i/\Delta_i, (\tilde{p}_i, 0))$ is a smooth irreducible surface, which shows that l is dimensionally proper w.r.t. E_i. Next, let us consider $\pi : \mathcal{X} \to \Delta$, $\tilde{p}_1, \ldots, \tilde{p}_8 : \Delta \to \mathcal{X}$, the versal deformation of $(\Gamma, p_1, \ldots, p_8)$. We have to prove that

$$\text{dim } \mathcal{G}^2_{15}(\mathcal{X}/\Delta, (\tilde{p}_i, (1,1,1))) = \text{dim } \Delta + \rho(l_\Gamma, \alpha^{l_\Gamma}(p_i)) = 35.$$

There is an isomorphism over Δ,

$$\mathcal{G}^2_{15}(\mathcal{X}/\Delta, (\tilde{p}_i, (1,1,1))) \simeq \mathcal{G}^2_7(\mathcal{X}/\Delta, (\tilde{p}_i, 0)).$$

If $\pi_0 : \mathcal{C} \to \mathcal{M}$ is the versal deformation space of Γ, then we denote by $\mathcal{G}^2_7 \to \mathcal{M}$ the scheme parametrizing $\mathfrak{g}^2_7$'s on curves of genus 15 'nearby' Γ (See Section 3 for this notation). Clearly $\mathcal{G}^2_7(\mathcal{X}/\Delta, (\tilde{p}_i, 0)) \simeq \mathcal{G}^2_7 \times_{\mathcal{M}} \Delta$, so it suffices to prove that $\mathcal{G}^2_7$ has the expected dimension at the point $(\Gamma, \mathfrak{g}^2_7)$. For this we use Prop. 3.1. We have that $N_{\Gamma/\mathbb{P}^2} = \mathcal{O}_\Gamma(7), K_\Gamma = \mathcal{O}_\Gamma(4)$, hence

$$H^1(\Gamma, N_{\Gamma/\mathbb{P}^2}) \simeq H^0(\Gamma, \mathcal{O}_\Gamma(-3))^\vee = 0,$$

so l is dimensionally proper w.r.t. Γ as well. We conclude that l is smoothable.

Step 3) *There exists a smoothable limit* $\mathfrak{g}^3_{20}$ *on* X, *that is* $[X] \in \overline{\mathcal{M}}^3_{20}$. First we notice that there is an isomorphism $\Gamma \xrightarrow{\sim} G^1_6(\Gamma)$, given by

$$\Gamma \ni p \mapsto |\mathfrak{g}^2_7 - p| \in G^1_6(\Gamma).$$

Consequently, there is a 2-dimensional family of $\mathfrak{g}^3_{12}$'s on Γ, of the form $\mathfrak{g}^3_{12} = \mathfrak{g}^1_6 + \mathfrak{h}^1_6 = |2\mathfrak{g}^2_7 - p - q|$, where $p, q \in \Gamma$. Pick $l_0 = l_0' + l_0''$, with $l_0', l_0'' \in G^1_6(\Gamma)$, a general $\mathfrak{g}^3_{12}$ of this type.

We construct l, a limit $\mathfrak{g}^3_{20}$ on X, as follows: the Γ-aspect is given by $l_\Gamma = l_0(p_1+\cdots p_8)$, and because of the generality of the chosen l_0 we have that $\rho(l_\Gamma, \alpha^{l_\Gamma}(p_1), \ldots, \alpha^{l_\Gamma}(p_8)) = -9$. The E_i-aspect is given by $l_{E_i} = \mathfrak{g}^3_4(16p_i)$, where $\mathfrak{g}^3_4 = |3p_i + x_i|$, with $x_i \in E_i - \{p_i\}$, for $i = 1, \ldots, 8$. It is clear that $\rho(l_{E_i}, \alpha^{l_{E_i}}(p_i)) = 1$ and that $l' = \{l_\Gamma, l_{E_i}\}$ is a refined limit $\mathfrak{g}^3_{20}$ on X.

In order to prove that l' is dimensionally proper, we first notice that l' is dimensionally proper w.r.t. the elliptic tails E_i. We now prove that l' is dimensionally proper w.r.t. Γ. As in the previous step, we consider $\pi : \mathcal{X} \to \Delta,\ \tilde{p}_1, \ldots, \tilde{p}_8 : \Delta \to \mathcal{X}$, the versal deformation of $(\Gamma, p_1, \ldots, p_8)$ and $\pi_0 : \mathcal{C} \to \mathcal{M}$, the versal deformation space of Γ. There is an isomorphism over Δ

$$\mathcal{G}^3_{20}(\mathcal{X}/\Delta, (\tilde{p}_1, \alpha^{l_\Gamma}(p_1), \ldots, (\tilde{p}_8, \alpha^{l_\Gamma}(p_8))) \simeq \mathcal{G}^3_{12}(\mathcal{C}/\mathcal{M}) \times_{\mathcal{M}} \Delta.$$

It suffices to prove that $\mathcal{G}^3_{12} = \mathcal{G}^3_{12}(\mathcal{C}/\mathcal{M})$ has a component of the expected dimension passing through (Γ, l_0). In this way, the genus 23 problem is turned into a deformation theoretic problem in genus 15. Denote as usual by $\sigma : \mathcal{G}^3_{12} \to \mathcal{M}$ the natural projection. According to Prop.3.1, it will be enough to exhibit an element $(C, l) \in \mathcal{G}^3_{20}$, sitting in the same component as (Γ, l_0), such that the linear system l is base point free and simple, and if $\phi : C \to \mathbb{P}^3$ is the map induced by l, then $H^1(C, N_\phi) = 0$. Certainly we cannot take C to be a smooth plane septic because in this case $H^1(C, N_\phi) \neq 0$, as one can easily see. Instead, we consider the 6-gonal locus in a neighbourhood of the point $[\Gamma] \in \mathcal{M}_{15}$, or equivalently, the 6-gonal locus in $\mathcal{M}$, the versal deformation space of Γ. One has the projection $\mathcal{G}^1_6 \to \mathcal{M}$, and the scheme $\mathcal{G}^1_6$ is smooth (and irreducible) of dimension $37 (= 2g + 2d - 5; g = 15, d = 6)$. We denote by

$$\mu : \mathcal{G}^1_6 \times_{\mathcal{M}} \mathcal{G}^1_6 \to \mathcal{M}, \quad \mu([C, l, l']) = [C].$$

Let us pick a component $\mathcal{X} \subseteq \mathcal{G}^1_6 \times_{\mathcal{M}} \mathcal{G}^1_6$ such that $(\Gamma, l'_0, l''_0) \in \mathcal{X}$. The general point of $\mathcal{X}$ corresponds to a curve C with two base-point-free pencils $l', l'' \in G^1_6(C)$ such that if $f' : C \to \mathbb{P}^1$ and $f'' : C \to \mathbb{P}^1$ are the corresponding morphisms, then

$$\phi = (f', f'') : C \to \mathbb{P}^1 \times \mathbb{P}^1$$

is birational. We denote by $\eta : \mathcal{X} \to \mathcal{G}^3_{12}$, the map given by $\eta(C, l', l'') := (C, l' + l'')$. The fact that η is well-defined follows from the base-point-free-pencil-trick. There is a stratification of $\mathcal{M}$ given by the number of pencils: for $i \geq 0$ we define,

$$\mathcal{M}(i)^0 := \{[C] \in \mathcal{M} : C \text{ possesses } i \text{ mutually independent, base-point-free } \mathfrak{g}^1_6\text{'s }\},$$

and $\mathcal{M}(i) := \overline{\mathcal{M}(i)^0}$. The strata $\mathcal{M}(i)^0$ are constructible subsets of $\mathcal{M}$, the first stratum $\mathcal{M}(1) = \text{Im }(\mathcal{G}^1_6)$ is just the 6-gonal locus; the stratum $\mathcal{M}(2)$ is irreducible and $\dim \mathcal{M}(2) = g + 4d - 7 = 32$ (cf. [AC1]). We denote by $\mathcal{M}_{sept} := \overline{m(U_{7,15}) \cap \mathcal{M}}$, the closure of the locus of smooth plane septics in $\mathcal{M}$, and by $\mathcal{M}_{oct} := \overline{m(U_{8,15}) \cap \mathcal{M}}$, the locus of curves which are normalizations of plane octics with 6 nodes. Since the Severi varieties $U_{7,15}$ and $U_{8,15}$ are irreducible, so are the loci $\mathcal{M}_{sept}$ and $\mathcal{M}_{oct}$. Furthermore $\dim \mathcal{M}_{sept} = 27$ and $\dim \mathcal{M}_{oct} = 30$. We prove that $\mathcal{M}_{sept} \subseteq \mathcal{M}_{oct}$, hence $\mathcal{M}_{oct} \subseteq \mu(\mathcal{X})$. Indeed, let us pick $Y \subseteq \mathbb{P}^2$ a smooth plane septic, and $L \subseteq \mathbb{P}^2$ a general

line, $L.Y = p_1 + \cdots + p_7$. Denote $X := C \cup L$, deg $(X) = 8, p_a(X) = 21$. We consider the node p_7 unassigned, while $p_1, \ldots p_6$ are assigned. By using [Ta] Theorem 2.13, there exists a flat family of plane curves $\pi : \mathcal{X} \to B$ and a point $0 \in B$, such that $X_0 = \pi^{-1}(0) = X$, while for $0 \neq b \in B$, the fibre X_b is an irreducible octic with nodes $p_1(b), \ldots p_6(b)$, and such that $p_i(b) \to p_i$, when $b \to 0$, for $i = 1, \ldots, 6$. If $\mathcal{X}' \to B$ is the family resulting by normalizing the surface $\mathcal{X}$, and $\eta : \mathcal{X}'' \to B$ is the stable family associated to the semistable family $\mathcal{X}' \to B$, then we get that $\eta^{-1}(0) = Y$, while $\eta^{-1}(b)$ is the normalization of X_b for $b \neq 0$. This proves our contention.

We are going to show that given a general point $[C] \in \mathcal{M}_{oct}$, and $(C, l, l') \in \mu^{-1}([C])$, the condition $H^1(C, N_\phi) = 0$ is satisfied, hence $\mathcal{G}^3_{12}$ is smooth of the expected dimension at the point $(C, l + l')$. This will prove the existence of a component of $\mathcal{G}^3_{12}$ passing through (Γ, l_0) and having the expected dimension. We take $\overline{C} \subseteq \mathbb{P}^2$, a general point of $U_{8,15}$, with nodes $p_1, \ldots, p_6 \in \mathbb{P}^2$ in general position. Theorem 3.2 from [AC1] ensures that there exists a plane octic having 6 prescribed nodes in general position. Let $\nu : C \to \overline{C}$ be the normalization map, $\nu^{-1}(p_i) = q_i' + q_i''$ for $i = 1, \ldots, 6$. Choose two nodes, say p_1 and p_2, and denote by $\mathfrak{g}^1_6 = |H - q_1' - q_1''|$ and $\mathfrak{h}^1_6 = |H - q_2' - q_2''|$, the linear series obtained by projecting $\overline{C}$ from p_1 and p_2 respectively. Here $H \in |\nu^* \mathcal{O}_{\mathbb{P}^2}(1)|$ is an arbitrary line section of C. The morphism induced by $(\mathfrak{g}^1_6, \mathfrak{h}^1_6)$ is denoted by $\phi : C \to \mathbb{P}^1 \times \mathbb{P}^1$, and $\phi_1 = s \circ \phi : C \to \mathbb{P}^3$, with $s : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ the Segre embedding. There is an exact sequence over C

$$0 \longrightarrow N_\phi \longrightarrow N_{\phi_1} \longrightarrow \phi^* N_{\mathbb{P}^1 \times \mathbb{P}^1/\mathbb{P}^3} \longrightarrow 0. \tag{11}$$

We can argue as in [AC2] p.473, that for a general $(C, \mathfrak{g}^1_6, \mathfrak{h}^1_6)$ with $[C] \in \mathcal{M}_{oct}$, we have $h^1(C, N_\phi) = 0$. Indeed, let us denote by $\mathcal{X}_0$ the open set of $\mathcal{X}$ corresponding to points (X, l, l') such that $\chi : X \to \mathbb{P}^1 \times \mathbb{P}^1$, the morphism associated to the pair of pencils (l, l') is birational, and by $\mathcal{U} \subseteq \mathcal{X}_0$ the variety of those points $(X, l, l') \in \mathcal{X}_0$ such that $H^1(X, N_\chi) \neq 0$. Define

$$\mathcal{V} := \{x = (X, l, l', \mathcal{F}, \mathcal{F}') : (X, l, l') \in \mathcal{U},\ \mathcal{F} \text{ is a frame for } l,\ \mathcal{F}' \text{ is a frame for } l'\}.$$

We may assume that for a generic $x \in \mathcal{U}$, the corresponding pencils l and l' are base-point-free. Suppose that $\mathcal{U}$ has a component of dimension α. For any $x \in \mathcal{V}$,

$$T_x(\mathcal{V}) \subseteq H^0(X, N_\chi), \text{ and } \dim T_x(\mathcal{V}) \geq \alpha + 2 \dim PGL(2) = \alpha + 6.$$

If $\mathcal{K}_\chi$ is the cuspidal sheaf of χ and $N_\chi' = N_\chi / \mathcal{K}_\chi$, then according to [AC1] Lemma 1.4, for a general point $x \in \mathcal{V}$ one has that,

$$T_x(\mathcal{V}) \cap H^0(X, \mathcal{K}_\chi) = \emptyset,$$

from which it follows that $\alpha \leq g - 6$. If not, one would have that $h^0(X, N_\chi') \geq g + 1$, and therefore by Clifford's Theorem, $h^1(X, N_\chi) = h^1(X, N_\chi') = 0$, which contradicts the definition of $\mathcal{U}$. Since clearly dim $\mathcal{M}_{oct} > g - 6$, we can assume that $h^1(C, N_\phi) = 0$, for the general $[C] \in \mathcal{M}_{oct}$. Therefore, by taking cohomology in (11), we get that

$$H^1(C, N_{\phi_1}) = H^1(C, \mathcal{O}_C(2)),$$

where $\mathcal{O}_C(1) = \phi_1^*\mathcal{O}_{\mathbb{P}^3}(1)$. By Serre duality,

$$H^1(C, \mathcal{O}_C(2)) = 0 \Longleftrightarrow |K_C - 2\mathfrak{g}^1_6 - 2\mathfrak{h}^1_6| = \emptyset. \tag{12}$$

Since $K_C = 5H - \sum_{i=1}^6 (q'_i + q''_i)$, equation (12) becomes

$$|H + q'_1 + q''_1 + q'_2 + q''_2 - \sum_{i=3}^{6} (q'_i + q''_i| = \emptyset. \tag{13}$$

If $L = \overline{p_1 p_2} \subseteq \mathbb{P}^2$, we can write $\nu^*(L) = q'_1 + q''_1 + q'_2 + q''_2 + x + y + z + t$, and (13) is rewritten as

$$|2H - x - y - z - t - \sum_{i=3}^{6} (q'_i + q''_i)| = \emptyset.$$

So, one has to show that there are no conics passing through the nodes p_3, p_4, p_5 and p_6 and also through the points in $L.\overline{C} - 2p_1 - 2p_2$. Because $[\overline{C}] \in U_{8,15}$ is general we may assume that x, y, z and t are distinct, smooth points of $\overline{C}$. Indeed, if the divisor $x+y+z+t$ on $\overline{C}$ does not consist of distinct points, or one of its points is a node, we obtain that $\overline{C}$ has intersection number 8 with the line L at 5 points or less. But according to [DH], the locus in the Severi variety

$$\{[X] \in U_{d,g} : X \text{ has total intersection number } m+3 \text{ with a line at } m \text{ points } \}$$

is a divisor on $U_{d,g}$, so we may assume that $[\overline{C}]$ lies outside this divisor. Now, if x, y, z and t are distinct and smooth points of $\overline{C}$, a conic satisfying (13) would necessarily be a degenerate one, and one gets a contradiction with the assumption that the nodes $p_1, \ldots, p_6$ of $\overline{C}$ are in general position. □

Remark: We have a nice geometric characterization of some of the strata $\mathcal{M}_i$. First, by using Zariski's Main Theorem for the birational projection $\mathcal{G}^1_6 \to \mathcal{M}(1)$, one sees that $[C] \in \mathcal{M}(1)_{sing}$ if and only if either $[C] \in \mathcal{M}(2)^0$, or C possesses a $\mathfrak{g}^1_6$ such that $\dim |2\mathfrak{g}^1_6| \geq 3$. In the latter case, the $\mathfrak{g}^1_6$ is a specialization of 2 different $\mathfrak{g}^1_6$'s in some family of curves, hence $\mathcal{M}(2) = \mathcal{M}(1)_{sing}$ (cf [Co2]). As a matter of fact, Coppens has proved that for $4 \leq k \leq [(g+1)/2]$ and $8 \leq g \leq (k-1)^2$, there exists a k-gonal curve of genus g carrying exactly 2 linear series $\mathfrak{g}^1_k$, so the general point of $\mathcal{M}(2)$ corresponds to a curve C of genus 15, having exactly 2 base-point-free $\mathfrak{g}^1_6$'s. Furthermore, using Coppens' classification of curves having many pencils computing the gonality (see [Co1]), we have that $\mathcal{M}(6) = \mathcal{M}_{oct}$, and $\mathcal{M}(i) = \mathcal{M}_{sept}$, for each $i \geq 7$.

Now we are in a position to complete the proof of Theorem 1:

Proof of Theorem 1 According to (2), it will suffice to prove that there exists a smooth curve $[Y] \in \mathcal{M}_{23}$ which carries a $\mathfrak{g}^3_{20}$, a $\mathfrak{g}^2_{17}$ but has no $\mathfrak{g}^1_{12}$'s. In the proof of Prop.5.4 we constructed a stable curve of compact type $[X] \in \overline{\mathcal{M}}_{23}$ such that $[X] \in \overline{\mathcal{M}}^2_{17} \cap \overline{\mathcal{M}}^3_{20}$, but $[X] \notin \overline{\mathcal{M}}^1_{12}$. If we prove that $[X] \in \overline{\mathcal{M}^2_{17} \cap \mathcal{M}^3_{20}}$, that is, there are smoothings of X which preserve both the $\mathfrak{g}^2_{17}$ and the $\mathfrak{g}^3_{20}$, we are done. One can write $\overline{\mathcal{M}}^2_{17} \cap \overline{\mathcal{M}}^3_{20} = Y_1 \cup \ldots \cup Y_s$, where Y_i are irreducible codimension 2 subvarieties of $\overline{\mathcal{M}}_{23}$. Assume that $[X] \in Y_1$. If $Y_1 \cap \mathcal{M}_{23} \neq \emptyset$, then $[X] \in Y_1 = \overline{Y_1 \cap \mathcal{M}_{23}} \subseteq \overline{\mathcal{M}^2_{17} \cap \mathcal{M}^3_{20}}$, and the conclusion follows.

So we may assume that $Y_1 \subseteq \overline{\mathcal{M}}_{23} - \mathcal{M}_{23}$. Because $[X] \in \Delta_1 - \bigcup_{j\neq 1}\Delta_j$, we must have $Y \subseteq \Delta_1$. It follows that $\overline{\mathcal{M}}^2_{17} \cap \Delta_1$ and $\overline{\mathcal{M}}^3_{20} \cap \Delta_1$ have Y_1 as a common component. According to Prop.4.2, both intersections $\overline{\mathcal{M}}^2_{17} \cap \Delta_1$ and $\overline{\mathcal{M}}^3_{20} \cap \Delta_1$ are irreducible, hence $\overline{\mathcal{M}}^2_{17} \cap \Delta_1 = \overline{\mathcal{M}}^3_{20} \cap \Delta_1 = Y_1$, which contradicts Prop.5.3. Theorem 1 now follows. □

6 The slope conjecture and $\mathcal{M}_{23}$

In this final section we explain how the slope conjecture in the context of $\mathcal{M}_{23}$ implies that $\kappa(\mathcal{M}_{23}) = 2$, and then we present evidence for this.

The slope of $\overline{\mathcal{M}}_g$ is defined as $s_g := \inf\{a \in \mathbb{R}_{>0} : |a\lambda - \delta| \neq \emptyset\}$, where $\delta = \delta_0 + \delta_1 + \cdots + \delta_{[g/2]}, \lambda \in \mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{R}$. Since λ is big, it follows that $s_g < \infty$. If $\mathbb{E}$ is the cone of effective divisors in $\mathrm{Div}(\overline{\mathcal{M}}_g) \otimes \mathbb{R}$, we define the slope function $s : \mathbb{E} \to \mathbb{R}$ by the formula

$$s_D := \inf\{a/b : a, b > 0 \text{ such that } \exists c_i \geq 0 \text{ with } [D] = a\lambda - b\delta - \sum_{i=0}^{[g/2]} c_i\delta_i\},$$

for an effective divisor D on $\overline{\mathcal{M}}_g$. Clearly $s_g \leq s_D$ for any $D \in \mathbb{E}$. When $g+1$ is composite, we obtain the estimate $s_g \leq 6 + 12/(g+1)$ by using the Brill-Noether divisors $\overline{\mathcal{M}}^r_{g,d}$, with $\rho(g, r, d) = -1$.

Conjecture 1 ([HMo]) *We have that $s_g \geq 6 + 12/(g+1)$ for each $g \geq 3$, with equality when $g+1$ is composite.*

Harris and Morrison also stated (in a somewhat vague form) that for composite $g+1$, the Brill-Noether divisors not only minimize the slope among all effective divisors, but they also single out those irreducible $D \in \mathbb{E}$ with $s_D = s_g$.

The slope conjecture has been proved for $3 \leq g \leq 11, g \neq 10$ (cf. [HMo], [CR3,4], [Tan]). A strong form of the conjecture holds for $g = 3$ and $g = 5$: on $\overline{\mathcal{M}}_3$ the only irreducible divisor of slope $s_3 = 9$ is the hyperelliptic divisor, while on $\overline{\mathcal{M}}_5$ the only irreducible divisor of slope $s_5 = 8$ is the trigonal divisor (cf. [HMo]). Conjecture 1 would imply that $\kappa(\mathcal{M}_g) = -\infty$ for all $g \leq 22$. For $g = 23$, we rewrite (1) as

$$nK_{\overline{\mathcal{M}}_{23}} = \frac{n}{c_{23,r,d}}[\overline{\mathcal{M}}^r_{g,d}] + 8n\,\delta_1 + \sum_{i=2}^{11}\frac{(i(23-i)-4)}{2}n\,\delta_i \quad (n \geq 1), \tag{14}$$

(see Section 2 for the coefficients $c_{g,r,d}$). As Harris and Morrison suggest, we can ask the question whether the class of any $D \in \mathbb{E}$ with $s_D = s_g$ is (modulo a sum of boundary components Δ_i) proportional to $[\overline{\mathcal{M}}^r_{23,d}]$, and whether the sections defining (multiples of) $\overline{\mathcal{M}}^r_{23,d}$ form a maximal algebraically independent subset of the canonical ring $R(\overline{\mathcal{M}}_{23})$. If so, it would mean that the boundary divisor $8n\delta_1 + (1/2)\sum_{i=2}^{11} n(i(23-i)-4)\delta_i$ is a fixed part of $|nK_{\overline{\mathcal{M}}_{23}}|$. Moreover, using our independence result for the three Brill-Noether divisors, it would follow that $h^0(\overline{\mathcal{M}}_{23}, nK_{\overline{\mathcal{M}}_{23}})$ grows quadratically in n, for n sufficiently high and sufficiently divisible, hence $\kappa(\mathcal{M}_{23}) = 2$. We would also have that $\Sigma \cap \mathcal{M}_{23} = \mathcal{M}^1_{12} \cap \mathcal{M}^2_{17} \cap \mathcal{M}^3_{20}$, with Σ the common base locus of all the linear systems

$|nK_{\overline{\mathcal{M}}_{23}}|$.

Evidence for these facts is of various sorts: first, one knows (cf. [Ta], [CR3]) that $|nK_{\overline{\mathcal{M}}_{23}}|$ has a large fixed part in the boundary: for each $n \geq 1$, every divisor in $|nK_{\overline{\mathcal{M}}_{23}}|$ must contain Δ_i with multiplicity $16n$ when $i = 1$, $19n$ when $i = 2$, and $(21 - i)n$ for $i = 3, \ldots, 9$ or 11. The results for Δ_1 and Δ_2 are optimal since these multiplicities coincide with those in (14). Note that $[\Delta_1] = 2\delta_1$.

Next, one can show that certain geometric loci in $\mathcal{M}_{23}$ which are contained in all three Brill-Noether divisors, are contained in Σ as well. The method is based on the trivial observation that for a family $f : X \to B$ of stable curves of genus 23 with smooth general member, if $B.K_{\overline{\mathcal{M}}_{23}} < 0$ (or equivalently, slope$(X/B) = \delta_B/\lambda_B > 13/2$), then $\phi(B) \subseteq \Sigma$, where $\phi : B \to \overline{\mathcal{M}}_{23}, \phi(b) = [X_b]$, is the associated moduli map. We have that:

• One can fill up the d-gonal locus $\overline{\mathcal{M}}_d^1$ with families $f : X \to B$ of stable curves of genus g such that slope(X/B) is $8+4/g$ in the hyperelliptic case, $36(g+1)/(5g+1)$ in the trigonal case, and $4(5g + 7)/(3g + 1)$ in the tetragonal case (cf. [Sta]). For $g = 23$ it follows that $\mathcal{M}_4^1 \subseteq \Sigma$. Note that this result is not optimal if we believe the slope conjecture since we know that $\mathcal{M}_8^1 \subseteq \mathcal{M}_{12}^1 \cap \mathcal{M}_{17}^2 \cap \mathcal{M}_{20}^3$. (The inclusion $\mathcal{M}_8^1 \subseteq \mathcal{M}_{20}^3$ is a particular case of a result from [CM].)

• We take a pencil of nodal plane curves of degree d with f assigned nodes in general position such that $\binom{d-1}{2} - f = 23$, and with b base points, where $4f + b = d^2$. After blowing-up the base points, we have a pencil $Y \to \mathbb{P}^1$ with fibre $[Y_t] \in \overline{\mathcal{M}}_d^2$. For this pencil $\lambda_{\mathbb{P}^1} = \chi(\mathcal{O}_Y) + 23 - 1 = 23$ and $\delta_{\mathbb{P}^1} = c_2(Y) + 88 = 91 + b + f$. The condition $\delta_{\mathbb{P}^1}/\lambda_{\mathbb{P}^1} > 13/2$ is satisfied precisely when $d \leq 10$, hence taking into account that such pencils fill up $\mathcal{M}_d^2$, we obtain that $\mathcal{M}_{10}^2 \subseteq \Sigma$. Note that $\mathcal{M}_{10}^2 \subseteq \mathcal{M}_8^1$, and as mentioned above, the 8-gonal locus is contained in the intersection of the Brill-Noether divisors.

• In a similar fashion we can prove that $\mathcal{M}_{23,\gamma}(2)$, the locus of curves of genus 23 which are double coverings of curves of genus γ is contained in Σ for $\gamma \leq 5$.

The fact that the slopes of other divisors on $\overline{\mathcal{M}}_{23}$ (or on $\overline{\mathcal{M}}_g$ for arbitrary g) consisting of curves with special geometric characterization, are larger than $6 + 12/(g + 1)$, lends further support to the slope hypothesis. In another paper we will compute the class of various divisors on $\overline{\mathcal{M}}_{23}$: the closure in $\overline{\mathcal{M}}_{23}$ of the locus

$$D_e := \{[C] \in \mathcal{M}_{23} : \text{there exists } l \in G^1_{(23+e)/2}(C) \text{ and } p \in C, \text{ such that } w^l(p) \geq e\},$$

for $3 \leq e \leq 19$ and e odd, and the closure of the locus

$$\{[C] \in \mathcal{M}_{23} : C \text{ has a } \mathfrak{g}^2_{18} \text{ with a 5-fold point, i.e. } \exists D \in C^{(5)} \text{ such that } \mathfrak{g}^2_{18}(-D) = \mathfrak{g}^1_{13}\}.$$

In each case we will show that the slope estimate holds.

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